Find closed forms for the following sums:

P.1.(40 pt)

$$\varphi(m) = \sum_{n=0}^{m-1} n^3(n+3)$$

P.2.(40 pt)

$$f(m) = \sum_{n=1}^{3m-1} \left\lfloor \frac{2n+1}{3} \right\rfloor$$

P.3.(40 pt)

$$g(m,c) = \sum_{n=0}^{m-1} n^2 \cdot n^{\underline{c}}$$

Solutions

P.1. First, use a difference scheme to calculate the differences $\Delta^0\psi(0), \ldots, \Delta^4\psi(0)$ for $\psi(n) = n^3(n+3)$:

$$\begin{array}{ccccccccc}
0 & & 4 & & 40 & & 162 & & 448 \\
 & 4 & & 36 & & 122 & & 286 & \\
 & & 32 & & 86 & & 164 & & \\
 & & & 54 & & 78 & & & \\
 & & & & 24 & & & &
\end{array}$$

Hence,

$$\psi(n) = \frac{4}{1!}n^{\underline{1}} + \frac{32}{2!}n^{\underline{2}} + \frac{54}{3!}n^{\underline{3}} + \frac{24}{4!}n^{\underline{4}} = 4n^{\underline{1}} + 16n^{\underline{2}} + 9n^{\underline{3}} + n^{\underline{4}} \ .$$

Therefore,

$$\begin{aligned}
\varphi(m) &= \sum\nolimits_0^m \psi(x)\delta x = 2m^{\underline{2}} + \frac{16}{3}m^{\underline{3}} + \frac{9}{4}m^{\underline{4}} + \frac{1}{5}m^{\underline{5}} \\
&= \frac{1}{5}m^5 + \frac{1}{4}m^4 - \frac{7}{6}m^3 + \frac{3}{4}m^2 - \frac{1}{30}m \ .
\end{aligned}$$

P.2. First, it is reasonable to try a difference scheme for $f(x)$ to calculate the differences $\Delta^0 f(0), \ldots, \Delta^4 f(0)$:

$$\begin{array}{ccccccccc}
0 & & 2 & & 10 & & 24 & & 44 \\
& 2 & & 8 & & 14 & & 20 & \\
& & 6 & & 6 & & 6 & & \\
& & & 0 & & 0 & & & \\
& & & & 0 & & & &
\end{array}$$

Hence, the difference scheme suggests that $f(x) = 2 \cdot \binom{m}{1} + 6 \cdot \binom{m}{2} = 3m^2 - m$. This can be proved by induction. Indeed, if the claim holds for m, then it also holds for $m+1$, because:

$$\begin{aligned}
f(m+1) &= \sum_{n=1}^{3(m+1)-1} \left\lfloor \frac{2n+1}{3} \right\rfloor \\
&= f(m) + \left\lfloor \frac{2(3m)+1}{3} \right\rfloor + \left\lfloor \frac{2(3m+1)+1}{3} \right\rfloor + \left\lfloor \frac{2(3m+2)+1}{3} \right\rfloor \\
&= 3m^2 - m + 2m + (2m+1) + (2m+1) \\
&= 3(m+1)^2 - (m+1) \ .
\end{aligned}$$

P.3. We use the observation that $n^2 = (n+2)(n+1) - 3(n+1) + 1$ and therefore:

$$\begin{aligned}
g(n,c) &= \sum_{n=0}^{m-1} \left[(n+2)^{\underline{c+2}} - 3(n+1)^{\underline{c+1}} + n^{\underline{c}} \right] \\
&= \sum_{0}^{m} \left[(x+2)^{\underline{c+2}} - 3(x+1)^{\underline{c+1}} + x^{\underline{c}} \right] \delta x \\
&= \frac{(m+2)^{\underline{c+3}}}{m+3} - 3\frac{(m+1)^{\underline{c+2}}}{c+2} + \frac{m^{\underline{c+1}}}{c+1} \\
&= c! \cdot \left[(c+1)(c+2)\binom{x+2}{c+3} - 3(c+1)\binom{x+1}{c+2} + \binom{x}{c+1} \right] \ .
\end{aligned}$$