

**P.1.(35 pt)** Let  $f(n)$  and  $g(n)$  be non-decreasing positive-valued functions such that

$$g(n) = O(f(n)) \quad \text{and} \quad f(n) \neq O(g(n)) .$$

Do these assumptions imply  $g(n) = o(f(n))$ ?

**P.2.(40 pt)** In the definition of Turing machines it is assumed that the program is finite and constant size. Show that if infinite programs were allowed then  $\mathbf{P} = \mathbf{NP}$ .

**P.3.(25 pt)** Let  $L_1 \in \mathbf{NP}$  and  $L_2 \in \mathbf{coNP}$ . What can you say about  $L_1 \setminus L_2$  and  $L_2 \setminus L_1$ ?

## Solutions

**P.1.** These assumptions do not imply  $g(n) = o(f(n))$ . Let  $f(n) = n$  and

$$g(n) = 2^{2^{\lfloor \log_2 \log_2 x \rfloor}}$$

On the one hand,  $g(n) \leq f(n)$  (which also implies  $g(n) = O(f(n))$ ) and

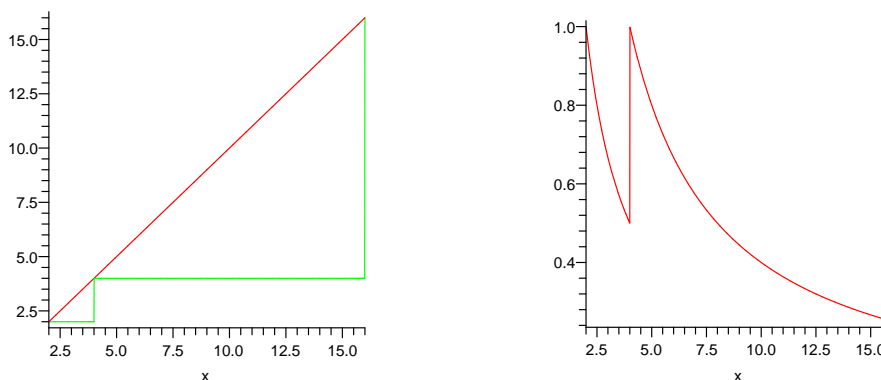


Figure 1: The graphs of functions  $f(x) = x$ ,  $g(x)$  (left) and  $g(x)/f(x)$  (right).

$f(n) \neq O(g(n))$  because there is no non-zero lower bound for the ratio  $g(x)/f(x)$ . Indeed for  $x_k = 2^{2^k} - 1$ , we have

$$\lim_{k \rightarrow \infty} \frac{g(x_k)}{f(x_k)} = \lim_{k \rightarrow \infty} \frac{2^{2^{k-1}}}{2^{2^k}} = \lim_{k \rightarrow \infty} 2^{2^{-k+1}} = 0 .$$

On the other hand, if we choose  $x'_k = 2^{2^k}$ , we have that

$$\lim_{k \rightarrow \infty} \frac{g(x'_k)}{f(x'_k)} = \lim_{k \rightarrow \infty} \frac{2^{2^k}}{2^{2^k}} = 1 ,$$

and therefore  $g(n) \neq o(f(n))$ .

**P.2.** For every non-deterministic Turing machine  $N$  we use a pre-computed table of output bits  $N(\perp), N(0), N(1), N(00), N(01), N(10), \dots$  (Figure 2). The running time of this infinite program when computing  $N(x)$  is  $O(|x|)$ .

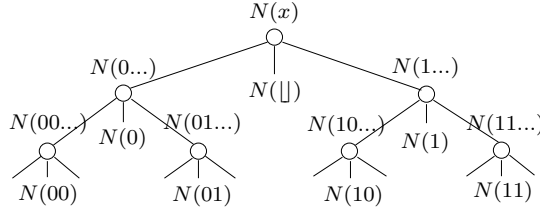


Figure 2: Infinite poly-time program for computing  $N(x)$ .

**P.3.**  $L_1 \setminus L_2 \in \mathbf{NP}$  and  $L_2 \setminus L_1 \in \mathbf{coNP}$ . First, it is easy to see that both  $\mathbf{NP}$  and  $\mathbf{coNP}$  are closed under the  $\cap$  operation. Indeed, if  $L, L' \in \mathbf{NP}$  then there are poly-time Turing machines  $M$  and  $M'$  such that

$$\begin{aligned} x \in L &\Leftrightarrow \exists a: M(x, a) = 1 ; \\ x \in L' &\Leftrightarrow \exists a: M'(x, a) = 1 . \end{aligned}$$

Now define a new poly-time machine  $M''(x, a||a') = M(x, a) \wedge M'(x, a')$ . We have that  $\exists a, a': M''(x, a||a') = 1$  if and only if  $x \in L \cap L'$ . Hence,  $L \cap L' \in \mathbf{NP}$ . If  $L, L' \in \mathbf{coNP}$  then there are poly-time Turing machines  $M$  and  $M'$  such that

$$\begin{aligned} x \in L &\Leftrightarrow \forall a: M(x, a) = 1 ; \\ x \in L' &\Leftrightarrow \forall a: M'(x, a) = 1 . \end{aligned}$$

Similarly,  $x \in L \cap L'$  if and only if  $\forall a, a': M''(x, a||a') = 1$ . Hence,  $L \cap L' \in \mathbf{coNP}$ .

The statements now follow from the observations that  $L_1 \setminus L_2 = L_1 \cap \overline{L_2}$ ,  $L_2 \setminus L_1 = L_2 \cap \overline{L_1}$ ,  $L_2 \in \mathbf{coNP} \Leftrightarrow \overline{L_2} \in \mathbf{NP}$ , and  $L_1 \in \mathbf{NP} \Leftrightarrow \overline{L_1} \in \mathbf{coNP}$ .