Introduction to Combinatorics and Cryptography
Lecture 1:
Direct Counting, Subsets, Permutations
Subsets of a Set

\( \mathbb{N} = \{0, 1, 2, \ldots \} \)—the set of all natural numbers.

There are \( 2^n \) subsets of an \( n \)-element set.

How many subsets does the set \( \{A, B, C\} \) have? (is it also true for \( n = 0 \))

How many subsets does a 100-element set have?

At a dinner, there are four choices for the appetizer, two choices for the main course, and three choices for the dessert. How many different meals are possible?


Principles of Direct Counting

Counting as one of the main disciplines in mathematics.

**Multiplication principle**: If $A$ can be done in $n$ different ways and $B$ can (independently) be done in $m$ different ways, then $A$ and $B$ together can be done in

$$n \cdot m$$

different ways.

**Addition principle**: If we can either do $A$ or $B$, where $A$ can be done in $n$ ways and $B$ can be done in $m$ ways, then we have

$$n + m$$

different ways to act.
Double Counting

Double-Counting Principle: If we count the same things in two different ways, we must have the same result.

This is used for proving identities of type:

\[ u(n, m) = v(n, m) , \]

where \( u \) and \( v \) are functions that convert natural numbers to natural numbers.
Permutations

\( P(n, k) \) — number of \( k \)-permutations of an \( n \)-element set.

By applying the multiplication principle, we get:

\[
P(n, k) = n(n - 1)(n - 2) \ldots (n - k + 1) = \frac{n!}{(n - k)!}
\]

\( P(n, n) = n! \) — a special case.

Alternative notion \( P(n, k) = n^k \) (\( k \)-th falling factorial power)

How many ways may 10 books be arranged on a shelf?

How many ways may four books from a set of 10 books be arranged on a shelf?
Combinations

\[ C(n, k) = \binom{n}{k} \] — the number of \( k \)-element subsets of an \( n \)-element set.

Prove by using double counting and the two direct counting principles:

\[
\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n , \\
\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k} \quad \text{for } n > 0
\]

Prove \( P(n, k) = C(n, k) \cdot k! \) via double counting. Imply that

\[
\binom{n}{k} = \frac{n^k}{k!} = \frac{n!}{k!(n-k)!} .
\]
Binomial Theorem

Prove by using direct counting that:

\[(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n.\]

In case \(n = 2\) (associating terms with subsets):

\[
\begin{align*}
(1 + x)(1 + x) &= 1 \cdot 1 + 1 \cdot x + x \cdot 1 + x \cdot x \\
&= 1 + x + x + x^2
\end{align*}
\]

or ... using \(x_1\) and \(x_2\) instead ... with assumption that \(x_1 = x_2 = x\)

\[(1 + x)^2 = (1 + x_1)(1 + x_2) = 1 + \frac{x_1 + x_2 + x_1 x_2}{\binom{2}{1}}.\]

\[(1 + x)^3 = (1 + x_1)(1 + x_2)(1 + x_3) = 1 + \frac{x_1 + x_2 + x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_2 x_3}{\binom{3}{1}} + \frac{x_1 x_2 x_3}{\binom{3}{2}}.\]
Lecture 2:
Binomial Coefficient Identities
Binomial Coefficient Identities

\[
\binom{n}{k} = \binom{n}{n-k},
\]
as to define a \(k\)-people team it is sufficient to name the \(n-k\) people who will not be in the team.

\[
\binom{n}{k}^k = n\binom{n-1}{k-1},
\]
because one may choose a \(k\)-people team with a captain by choosing the captain first (\(n\) possible choices) and then the rest of the team (\(k-1\) people from an \(n-1\) element set)

\[
\binom{n}{k}^j \binom{k}{j} = \binom{n}{j} \binom{n-j}{k-j} \quad (\text{for } 0 \leq j \leq k \leq n)
\]
as for choosing a \(k\)-element committee with a \(j\)-element subcommittee by first choosing the subcommittee and then the rest of the \(k-j\) members.
\[
\binom{m+n}{k} = \sum_{i=1}^{k} \binom{m}{i} \binom{n}{k-i} \quad \text{(for } m, n, k \geq 0 \text{)},
\]
as for choosing a team with \( k \) people from \( m \) boys and \( n \) girls we have, on one hand, \( \binom{m+n}{k} \) possibilities. On the other hand, for a fixed number \( i \) (where \( 0 \leq i \leq k \)) of boys in the team, we have \( \binom{m}{i} \) possibilities to choose the boys and \( \binom{n}{k-i} \) possibilities to choose the girls. As a corollary:

\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n},
\]
because \( \binom{n}{k} = \binom{n}{n-k} \).
Odd and Even Subsets

To show that for \( n > 0 \) there are always the same number of odd and even subsets, we compute by using the Binomial theorem

\[
0 = (1 - 1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k
= \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots - \binom{n}{1} - \binom{n}{3} - \binom{n}{5} - \ldots
\]

even subsets odd subsets
Using Derivatives

\[ \sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}, \]

because for choosing a non-empty team with a captain one may choose the captain first and then an arbitrary subset of an \((n - 1)\)-element set.

We may prove that same thing from \((1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\) by applying \(\frac{d}{dx}\) to both sides:

\[ n(1 + x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k x^{n-1}, \]

and taking \(x = 1\).
Generalized Binomial Coefficients

For every real \( \alpha \), define \( \binom{\alpha}{0} = 1 \) and

\[
\binom{\alpha}{k} = \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)}{k!} = \frac{\alpha^k}{k!}.
\]

Binomial Theorem:

\[
(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k
\]

for every \( |x| < 1 \).
Corollaries

\[
\binom{-n}{k} = \frac{(-n)(-n-1)(-n-2) \ldots (-n-k+1)}{k!} = (-1)^k \binom{n+k-1}{k}
\]

\[
\frac{1}{(1+x)^n} = (1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k
\]

\[
\frac{1}{(1-x)^n} = \frac{1}{(1+(-x))^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k
\]
Axiomatic Theory of Natural Numbers

In 1889, Giuseppe Peano (1858–1932) presented an axiomatic theory of natural numbers (0, 1, 2, . . .), using the next natural number function $n \mapsto n^+$ (where $n^+ = n + 1$) as the fundamental relation that satisfies the following axioms:
Axiom 1: $n \mapsto n^+$ is an injective function, i.e. every natural number $n$ has one and only one successor $n^+$, and if $n^+ = m^+$, then $n = m$.

Axiom 2: There is one and only one $m$, such that $n^+ \neq m$ for any $n$. Such $m$ is called zero and is denoted by 0.

Axiom 3 (Induction): If 0 has a property $A$ and (for every $n$) if $n$ has $A$ then also $n^+$ has $A$, then all natural numbers have the propery $A$:

$$A(0) \land \forall n: A(n) \rightarrow A(n^+) \Rightarrow \forall n: A(n).$$
Addition and its properties

Define addition as follows:

\[ x + 0 = x \]
\[ x + y^+ = (x + y)^+ \]

Addition is commutative \((x + y = y + x)\) because:

**Lemma 1:** \(\forall x: 0 + x = x = x + 0.\)

**Proof:** As \(0 + 0 = 0\) and if \(0 + x = x\), then

\[ 0 + x^+ = (0 + x)^+ = x^+ = x^+ + 0. \]

**Lemma 2:** If \(\forall x: y + x = x + y\), then \(\forall x: x + y^+ = y^+ + x.\)

**Proof:** \(0 + y^+ = y^+ + 0\) and if \(x + y^+ = y^+ + x\), then

\[ x^+ + y^+ = (x^+ + y)^+ = (y + x^+)^+ = (y + x)^++ = (x + y)^++ = (x + y^+)^+ = (y^+ + x)^+ = y^+ + x^+. \]
Addition is associative because:

1: \[(x + y) + 0 = x + y = x + (y + 0)\]

2: If \((x + y) + z = x + (y + z)\), then

\[
(x + y) + z^+ = ((x + y) + z)^+ = (x + (y + z))^+ = x + (y + z)^+ \\
= x + (y + z^+) .
\]

**Exercise:** Define multiplication as follows

\[
x \cdot 0 = 0 \\
x \cdot y^+ = x \cdot y + x
\]

and prove its main properties in a similar way.
Lecture 3:
Sums and Recurrence Relations
The Tower of Hanoi

Objective: transfer the entire tower to one of the other pegs,
• moving only one disk at a time and
• never moving a larger one into a smaller.
The Tower of Hanoi: Recurrence Relation

$T_n$—the minimum number of moves that transfer $n$ disks to another peg.

$T_0 = 0$.

$T_n \leq 2T_{n-1} + 1$, because we can transfer $n - 1$ smaller disk to another peg with $T_{n-1}$ moves, then move the largest disk, and finally, move the $n - 1$ smaller disks onto the largest disk with $T_{n-1}$ moves.

$T_n \geq 2T_{n-1} + 1$, because for moving the largest disk, the smaller ones have to be moved to another peg, which takes at least $T_{n-1}$ moves, then we have to move the largest disk (1 move) and finally move the $n - 1$ smaller disks onto the largest, which again takes at least $T_{n-1}$ moves.

Hence, $T_n = 2T_{n-1} + 1$. 
The Tower of Hanoi: Solution

To solve the recurrence relation:

\[
T_0 = 0 \\
T_n = 2T_{n-1} + 1 , \text{ for } n > 0,
\]

we first study \( T(n) \) with small values of \( n \):

\( T(1) = 1, T(2) = 3, T(3) = 7, T(4) = 15, \text{ etc.} \)

It seems like \( T(n) = 2^n - 1 \), which can be proved by induction.

How to \textit{compute} this formula, without guessing it first?
Sums as Recurrent relations

The sum

$$S_n = \sum_{k=0}^{n} a_k$$

can also be presented as a recurrence relation:

$$S_0 = a_0$$
$$S_n = S_{n-1} + a_n, \text{ for } n > 0.$$
A Special Case: The Repertoire Method

Let us study the recurrence relation:

\[
R_0 = \alpha \\
R_n = R_{n-1} + \beta + \gamma n , \quad \text{for } n > 0.
\]

We see that \( R_1 = \alpha + \beta + \gamma, \ R_2 = \alpha + 2\beta + 3\gamma, \) etc.

It seems reasonable to search for a solution in the form:

\[
R_n = A(n)\alpha + B(n)\beta + C(n)\gamma ,
\]

where \( A, B, C \) are certain functions of \( n \).

The so-called \textit{repertoire method} tries to plug in simple functions like \( R_n = 1, \ R_n = n, \ R_n = n^2, \) etc. in order to find \( A(n), B(n), C(n), \) etc.
Finding $A(n), B(n), C(n)$

1. If $R_n = 1 = R_{n-1} + 0 + 0n$, then $\alpha = 1, \beta = \gamma = 0$ and hence:
   
   $$1 = R_n = A(n) \cdot 1 \quad \Rightarrow A(n) = 1.$$  

2. If $R_n = n = (n - 1) + 1 = R_{n-1} + 1 + 0n$, then $\alpha = 0, \beta = 1, \gamma = 0$ and hence:
   
   $$n = R_n = A(n) \cdot 0 + B(n) \cdot 1 + C(n) \cdot 0 = B(n) \quad B(n) = n.$$  

3. If $R_n = n^2 = (n - 1)^2 - 1 + 2n$, then $\alpha = 0, \beta = -1, \gamma = 2$ and hence:
   
   $$n^2 = -B(n) + 2C(n) = -n + 2C(n) \quad C(n) = \frac{n(n+1)}{2}.$$
Back to the Tower of Hanoi

To solve the recurrence relation:

\[
T_0 = 0 \\
T_n = 2T_{n-1} + 1, \quad \text{for } n > 0,
\]

We convert it first by assigning \( S_n = \frac{T_n}{2^n} \) to convert the relation to the form:

\[
S_0 = 0 \\
S_n = S_{n-1} + \frac{1}{2^n}, \quad \text{for } n > 0,
\]

Hence, \( S_n = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} \) and

\[
2S_n = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n-1}} = 1 + S_n - \frac{1}{2^n},
\]

and therefore \( S_n = 1 - \frac{1}{2^n} \), which implies \( T_n = 2^n S_n = 2^n - 1 \).
Exercise

Find a formula for:

\[ S_n = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 . \]

For that, we first extend the repertoire method for more general recurrence relations of type:

\[
\begin{align*}
R_0 &= \alpha \\
R_n &= R_{n-1} + \beta + \gamma n + \delta n^2 , \quad \text{for } n > 0.
\end{align*}
\]

and search for a general solution

\[ R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta . \]

As we already know that \( A(n) = 1, \ B(n) = n, \) and \( C(n) = \frac{n(n+1)}{2}, \) we only need to determine \( D(n). \)
For that, we plug in the function $R_n = n^3$ and get $0 = 0^3 = \alpha$ and:

$$n^3 = (n - 1)^3 + \beta + \gamma n + \delta n^2$$  \hspace{1cm} \text{(for all values of } n)$$

which implies $\beta = 1$, $\gamma = -3$, and $\delta = 3$. Hence,

$$n^3 = n - 3\frac{n(n + 1)}{2} + 3D(n),$$

which means that:

$$D(n) = \frac{1}{3} \left[ n^3 - n + \frac{3}{2}n(n + 1) \right] = \frac{1}{3}n(n + 1)(n + \frac{1}{2})$$

If we now take $R_n = 1^2 + 2^2 + \ldots + n^2$, it satisfies the general recurrence relations with $\alpha = \beta = \gamma = 0$ and $\delta = 1$. Hence,

$$1^2 + 2^2 + \ldots + n^2 = D(n) = \frac{1}{3}n(n + 1)(n + \frac{1}{2}).$$
Lecture 4:
Finite Calculus
Difference Operator

Infinite (ordinary) calculus uses *differential operator* $D$ that converts a function $f(x)$ to its derivative:

$$Df(x) = \lim_{h \to 0} \frac{f(x + h) - f(h)}{h}.$$ 

Finite calculus uses the *difference operator* $\Delta$ instead:

$$\Delta f(x) = f(x + 1) - f(x).$$
Derivative and Difference of a Power Function

Ordinary calculus uses the formula:

$$D(x^m) = mx^{m-1}.$$  

The difference $\Delta(x^m)$ is not that elegant, for example:

$$\Delta(x^3) = (x+1)^3 - x^3 = 3x^2 + 3x + 1.$$  

But there is another "power function" that behaves nicely: the falling factorial power $x^m$:

$$\Delta(x^m) = (x+1)^m - x^m$$

$$= (x+1)x(x-1)\ldots(x-m+2) - x(x-1)\ldots(x-m+2)(x-m+1)$$

$$= x(x-1)\ldots(x-m+2)[x+1-x+m-1] = mx^{m-1}.$$  

The “Indefinite Sum” Operator

The differential operator $D$ has inverse: the indefinite integral operator $\int$:

$$g(x) = Df(x) \iff \int g(x)\,dx = f(x) + C,$$

where $C$ is any constant. This is because $D(C) = 0$.

The difference operator $\Delta$ also has inverse: the indefinite sum operator $\sum$:

$$g(x) = \Delta f(x) \iff \sum g(x)\delta x = f(x) + C,$$

where $C$ is any 1-periodic function, i.e. $C(x + 1) = C(x)$, which implies

$$\Delta C(x) = C(x + 1) - C(x) = 0.$$
Properties of $\Sigma$

We want $\Sigma$ to satisfy the Newton-Leibniz formula:

$$\int_a^b g(x) \, dx = f(x) \big|_a^b = f(b) - f(a)$$

$$\sum_a^b g(x) \delta x = f(x) \big|_a^b = f(b) - f(a).$$

Assuming that $g(x) = \Delta f(x) = f(x + 1) - f(x)$ we have

$$\sum_a^a g(x) = f(a) - f(a) = 0$$

$$\sum_{a+1}^a g(x) = f(a + 1) - f(a) = \Delta f(a) = g(a).$$
Formula for $\sum_{a}^{b} g(x) \delta x$

\[
\begin{align*}
\sum_{a}^{b+1} g(x) \delta x - \sum_{a}^{b} g(x) \delta x &= [f(b + 1) - f(a)] - [f(b) - f(a)] \\
&= f(b + 1) - f(b) = g(b) . \\
\end{align*}
\]

Hence,

\[
\begin{align*}
\sum_{a}^{a} g(x) \delta x &= 0 \\
\sum_{a}^{b+1} g(x) \delta x &= \sum_{a}^{b} g(x) \delta x + g(b) ,
\end{align*}
\]

and hence, by induction:

\[
\begin{align*}
\sum_{a}^{b} g(x) \delta x &= \sum_{k=a}^{b-1} g(k) = g(a) + g(a + 1) + \ldots + g(b - 1) .
\end{align*}
\]
Summing the Powers via Finite Calculus

\[ \int_0^n x^m \, dx = \frac{n^{m+1}}{m+1} \]

\[ \sum_0^n x^m \delta x = \frac{n^{m+1}}{m+1} \cdot \]

For example,

\[ \sum_{k=0}^n k = \sum_{0}^{n+1} x^1 \delta x = \frac{x^2}{2} \bigg|_{0}^{n+1} = \frac{(n+1)^2}{2} = \frac{n(n+1)}{2} , \]

and as \( x^2 = x^2 + x^1 \), we have:

\[ \sum_{k=0}^n k^2 = \sum_{0}^{n+1} (x^2 + x^1) \delta x = \frac{(n+1)^3}{3} + \frac{(n+1)^2}{2} = \frac{n(n+1)(n+\frac{1}{2})}{3} . \]
Negative Powers

\[ x^3 = x(x - 1)(x - 2) \]
\[ x^2 = x(x - 1) \]
\[ x^1 = x \]
\[ x^0 = 1 \]
\[ x^{-1} = \frac{1}{x + 1} \]
\[ x^{-2} = \frac{1}{(x + 1)(x + 2)} \]
\[ x^{-3} = \frac{1}{(x + 1)(x + 2)(x + 3)} \]
\[ \Delta x^{-2} = \frac{1}{(x + 2)(x + 3)} - \frac{1}{(x + 1)(x + 2)} \]
\[ = \frac{x + 1 - (x + 3)}{(x + 1)(x + 2)(x + 3)} \]
\[ = -2x^{-3}. \]

If \( m \neq 0 \):

\[ \sum_{a}^{b} x^{m} \delta x = \frac{x^{m+1}}{m + 1} \bigg|_{a}^{b}. \]
The Case $m = 1$ and Harmonic Sums

For integration, we have:

$$\int_a^b x^{-1} \, dx = \ln x \bigg|_a^b.$$ 

For summation, first recall that:

$$x^{-1} = \frac{1}{x+1} = \Delta f(x) = f(x + 1) - f(x),$$

which gives $f(x + 1) = f(x) + \frac{1}{x+1}$, i.e.

$$f(x) = H_x = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{x}$$

for every positive integer $x$, where $H_x$ is the harmonic sum. Note that

$$H_x \approx \ln x + 0.577 + \frac{1}{2x}.$$
\[ e^x \] and \[ 2^x \]

We know from calculus that \( D(e^x) = e^x \), but for which function \( f(x) \)

\[ \Delta f(x) = f(x) \, . \]

From

\[ \Delta f(x) = f(x + 1) - f(x) = f(x) \]

we get \( f(x + 1) = 2f(x) \), which means that \( f(x) = 2^x \).
Differences of Products

\[ D(uv) = uDv + vDu \]
\[ \Delta(uv) = u\Delta v + v(x + 1)\Delta u \]

Because

\[ \Delta(u(x)v(x)) = u(x + 1)v(x + 1) - u(x)v(x) \]
\[ = u(x + 1)v(x + 1) - u(x)v(x + 1) + u(x)v(x + 1) - u(x)v(x) \]
\[ = u(x)\Delta v(x) + v(x + 1)\Delta u(x) \]
Summation by Parts

\[
\int u(x) Dv(x) \, dx = uv - \int v(x) Du(x) \, dx
\]

\[
\sum u(x) \Delta v(x) \delta x = uv - \sum v(x + 1) \Delta u(x) \delta x
\]

Example: To find \( \sum_{k=0}^{n} k2^k \), we first compute the indefinite sum:

\[
\sum x2^x \delta x = x2^x - \sum 2^{x+1} \delta x = x2^x - 2^{x+1} + C,
\]

where \( u(x) = x \) and \( \Delta v(x) = 2^x \) and hence \( \Delta u(x) = 1 \) and \( v(x) = 2^x \). Then,

\[
\sum_{k=0}^{n} k2^k = \sum_{0}^{n+1} x2^x \delta x = x2^x - 2^{x+1} \bigg|_{0}^{n+1}
\]

\[
= [(n + 1)2^{n+1} - 2^{n+2}] - [0 \cdot 2^0 - 2^1]
\]

\[
= (n - 1)2^{n+1} + 2.
\]
Higher Order Differences

\[ \Delta^2 f(x) = \Delta f(x + 1) - \Delta f(x) = f(x + 2) - 2f(x + 1) + f(x) \]
\[ \Delta^3 f(x) = f(x + 3) - 3f(x + 2) + 3f(x + 1) - f(x) \]
\[ \ldots \]
\[ \Delta^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x + k) \]

As for \( d > 0 \) we have \( \Delta \left( \frac{x}{d} \right) = \binom{x}{d-1} \) and \( \Delta \left( \frac{x}{0} \right) = 0 \), then

\[ \Delta^n \left( \frac{x}{d} \right) = \begin{cases} \binom{x}{d-n} & \text{if } n \leq d \\ 0 & \text{if } n > d \end{cases} \]
Newton Series

Any polynomial \( f(x) = a_dx^d + a_{d-1}x^{d-1} + \ldots + a_1x + a_0 \) can be expressed in the form:

\[
f(x) = b_dx^d + b_{d-1}x^{d-1} + \ldots + b_1x + b_0
\]

\[
= c_d\binom{x}{d} + c_{d-1}\binom{x}{d-1} + \ldots + c_1\binom{x}{1} + c_0
\]

where \( b_k = \frac{c_k}{k!} \). As it is easy to check that \( (\Delta^k f)(0) = c_k \), hence

\[
f(x) = f(0)^{\binom{x}{0}} + \Delta f(0)^{\binom{x}{1}} + \Delta^2 f(0)^{\binom{x}{2}} + \ldots + \Delta^d f(0)^{\binom{x}{d}}
\]

\[
= f(0)^{\binom{x}{0}}0! + \Delta f(0)^{\binom{x}{1}}1! + \Delta^2 f(0)^{\binom{x}{2}}2! + \ldots + \Delta^d f(0)^{\binom{x}{d}}d!
\]

This is analogous to the Taylor series in calculus.
Example

For \( f(x) = x^3 \), we first compute the higher order differences:

\[
\begin{align*}
  f(0) &= 0, & f(1) &= 1, & f(2) &= 8, & f(3) &= 27 \\
  \Delta f(0) &= 1, & \Delta f(1) &= 7, & \Delta f(2) &= 19 \\
  \Delta^2 f(0) &= 6, & \Delta^2 f(1) &= 12 \\
  \Delta^3 f(0) &= 6
\end{align*}
\]

And using the Newton expansion, we have:

\[
x^3 = 6{\binom{x}{3}} + 6{\binom{x}{2}} + 1{\binom{x}{1}} + 0{\binom{x}{0}}
\]

\[
= 6 \frac{x^3}{3!} + 6 \frac{x^2}{2!} + x^1
\]

\[
= x^3 + 3x^2 + x^1 .
\]