NP and NP-completeness

Slides based on *S.Aurora, B.Barak. Complexity Theory: A Modern Approach.*

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There is often a big difference between:

- solving a problem from scratch, and
- verifying a given solution.

Crossword puzzles is one example.

We define the complexity class $\mathbf{NP}$ that aims to capture the set of problems whose solutions can be efficiently verified.

The class $\mathbf{NP}$

Class of problems having efficiently verifiable solutions.

A decision problem/language is in $\mathbf{NP}$ if given an input x, we can easily verify that x is a YES instance of the problem (x is in the language) if we are given the polynomial-size solution for x, that certifies this fact.

Def: A language $L \subseteq \{0,1\}^*$ is in $\mathbf{NP}$ if there exists a polynomial p and a polynomial-time Turing machine M such that for every $x \in \{0,1\}^n$:

$$x \in L \;\Leftrightarrow\; \exists u \in \{0,1\}^{p(|x|)}\colon M(x,u) = 1 \;\;.$$

If $x \in L$ and $u \in \{0,1\}^{p(|x|)}$ satisfy $M(x,u) = 1$ then we call u a *certificate* (or a *witness*) for x (with respect to the language L and machine M).

Problems in $\mathbf{NP}$

Independent set: Given a graph G and a number k, decide if there is a k-size independent subset of vertices in G. The certificate is the list of k vertices forming an independent set.

Traveling salesman: Given a set of n nodes, $\binom{n}{2}$ numbers d_{ij} denoting the distances between all pairs of nodes, and a number k, decide if there is a closed circuit (i.e., a "salesman tour") that visits every node exactly once and has total length at most k. The certificate is the sequence of nodes in the tour.

Subset sum: Given a list of n numbers $A_1, \ldots, A_n$ and a number T, decide if there is a subset of the numbers that sums up to T. The certificate is the list of members in this subset.

Problems in $\mathbf{NP}$

Linear programming: Given a list of m linear inequalities with rational coefficients over n variables $u_1, \ldots, u_n$ (in the form $a_1u_1 + a_2u_2 + \ldots + a_nu_n \leq b$ for some coefficients $a_1, \ldots, a_n, b$), decide if there is an assignment of rational numbers to the variables $u_1, \ldots, u_n$ that satisfies all the inequalities. The certificate is the assignment.

Integer programming: Given a list of m linear inequalities with rational coefficients over n variables $u_1, \ldots, u_m$, find out if there is an assignment of integer numbers to $u_1, \ldots, u_n$ satisfying the inequalities. The certificate is the assignment.

Graph isomorphism: Given two $n \times n$ adjacency matrices M_1 and M_2, decide if M_1 and M_2 define the same graph, up to renaming of vertices. The certificate is the permutation $\pi : [n] \to [n]$, such that M_2 is equal to M_1 after reordering M_1s indices according to π.

Problems in $\mathbf{NP}$

Composite numbers: Given a number N decide if N is a composite (i.e., non-prime) number. The certificate is the factorization of N.

Factoring: Given three numbers N,L and U decide if N has a factor M in the interval $[L, U]$. The certificate is the factor M.

Connectivity: Given a graph G and two vertices s, t in G, decide if s is connected to t in G. The certificate is the path from s to t.

Relation between $\mathbf{NP}$ and $\mathbf{P}$

We have the following trivial relationships between $\mathbf{NP}$ and the classes $\mathbf{P}$ and $\mathbf{DTIME}(T(n))$:

Claim 2.3: $\mathbf{P} \subseteq \mathbf{NP} \subseteq \bigcup_{c>1} \mathbf{DTIME}(2^{n^c})$.

Proof: Suppose $L \in \mathbf{P}$ is decided in poly-time by M, i.e.

$$x \in L \Leftrightarrow M(x) = 1 \Leftrightarrow \exists u \in \{0,1\}^0 M(x,u) = 1 \ .$$

Hence, $L \in \mathbf{NP}$.

If $L \in \mathbf{NP}$ and M and, $p(n)$ are as in the definition of $\mathbf{NP}$, then we can decide L in time $2^{O(p(n))}$ by enumerating all possible u and using M to check whether u is a valid certificate for the input x. The machine accepts iff such a u is ever found. Since $p(n) = O(n^c)$ for some $c > 1$, then this machine runs in $2^{O(n^c)}$ time.

Non-deterministic Turing machines

The class $\mathbf{NP}$ can also be defined using non-deterministic Turing machines (NDTMs). The only differences between an NDTM and a TM are:
- NDTM has two transition functions δ_0 and δ_1.
- NDTM has a special state we denote by q_{accept}.
- NDTM makes (at each step) an arbitrary choice as to which of its two transition functions to apply.

We say that a NDTM N outputs 1 on a given input x if there is some sequence of these non-deterministic choices that would make N reach q_{accept} on input x. Otherwise, if every sequence of choices makes N halt without reaching q_{accept}, then we say that N outputs 0.

We say that N runs in $T(n)$ time if for every $x \in \{0,1\}^n$ and every sequence of choices, $M(x)$ reaches either the halting state or q_{accept} within $T(|\ x\ |)$ steps.

Alternative definition of $\mathbf{NP}$

Def: For every function $T: \mathbb{N} \to \mathbb{N}$ and $L \subseteq \{0,1\}^*$, we say that $L \in \mathbf{NTIME}(T(n))$ if there is a constant $c > 0$ and a $cT(n)$-time NDTM N such that for every $x \in \{0,1\}^n$: $x \in L \iff N(x) = 1$.

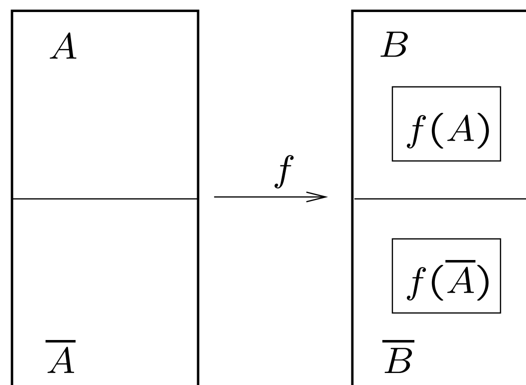
Theorem 2.6: $\mathbf{NP} = \cup_{c\in\mathbb{N}} \mathbf{NTIME}(n^c)$.

Proof idea: If L is decided by a $p(n)$-time NDTM N, then the sequence of choices that lead to q_{accept} can be used as a certificate of size $p(n)$.

If $L \in \mathbf{NP}$ (with machine M and cert-size $p(n)$) then we can construct a NDTM N that given $x \in \{0,1\}^n$ as input first makes $p(n)$ non-deterministic choices to write down $u \in \{0,1\}^{p(n)}$; after that, N computes $M(x,u)$ and finishes in state q_{accept} if $M(x,u) = 1$, otherwise N just halts.

Reducibility and NP-completeness

Def 2.7: A language $A \subseteq \{0,1\}^*$ is polynomial-time Karp reducible to a language $B \subseteq \{0,1\}^*$ denoted by $A \leq_p B$ if there is a poly-time computable function $f\colon \{0,1\}^* \to \{0,1\}^*$ such that for every $x \in \{0,1\}^*$:
$x \in A \;\Leftrightarrow\; f(x) \in B$.
We say that B is $\mathbf{NP}$-hard if $A \leq_p B$ for every $A \in \mathbf{NP}$. We say that B is $\mathbf{NP}$-complete if B is $\mathbf{NP}$-hard and $B \in \mathbf{NP}$.



Properties of $\leq_p$

Theorem 2.8:

- $A \leq_p B,\ B \leq_p C \ \Rightarrow \ A \leq_p C$.
- If A is $\mathbf{NP}$-hard and $A \in \mathbf{P}$ then $\mathbf{P} = \mathbf{NP}$.
- If A is $\mathbf{NP}$-complete, then $A \in \mathbf{P} \Leftrightarrow \mathbf{P} = \mathbf{NP}$.

Proof ideas: The main observation is that if p, q are two functions that have polynomial growth then their composition $p(q(n))$ also has polynomial growth.

If f_1 is a polynomial-time reduction from A to B and f_2 is a reduction from B to C then the mapping $x \mapsto f_2(f_1(x))$ is a polynomial-time reduction from A to C since $f_2(f_1(x))$ takes polynomial time to compute given x and $f_2(f_1(x)) \in C \Leftrightarrow x \in A$.

Do NP-complete languages exist?

It may not be clear that $\mathbf{NP}$ should possess a language that is as hard as any other language in the class. However, this does turn out to be the case:

Theorem 2.9: The following language is $\mathbf{NP}$-complete:

$$\mathbf{TMSAT} = \{\langle \alpha, x, 1^n, 1^t \rangle : \ \exists u \in \{0,1\}^n : 1 \leftarrow M_\alpha(x,u) \text{ within } t \text{ steps.}\}$$

where M_α denotes the TM represented by the string α.

The Cook-Levin Theorem shows that there are more natural examples of $\mathbf{NP}$-complete problems.

Boolean formulae and the CNF form.

Several $\mathbf{NP}$-complete problems come from propositional logic.

A *Boolean formula* over the variables $u_1, \ldots, u_n$ consists of the variables and the logical operators AND ($\wedge$), NOT ($\neg$) and OR ($\vee$). For example,

$$(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$$

is a Boolean formula that is True iff the majority of a, b, c are True.

If φ is a Boolean formula over variables $u_1, \ldots, u_n$, and $z \in \{0, 1\}^n$, then $\varphi(z)$ denotes the value of φ when the variables of φ are assigned the values z (1 means True and 0 means False).

A formula φ is *satisfiable* if there there exists some assignment z such that $\varphi(z)$ is True. Otherwise, we say that φ is unsatisfiable.

A Boolean formula over variables $u_1, \ldots, u_n$ is in CNF form (*Conjunctive Normal Form*) if it is an AND of ORs of variables or their negations. For example, the following is a 3CNF formula:

$$(u1 \vee \overline{u}_2 \vee u_3) \wedge (u_2 \vee \overline{u}_3 \vee u_4) \wedge (\overline{u}_1 \vee u_3 \vee \overline{u}_4) \ .$$

where $\overline{u}$ denotes the negation of u.

More generally, a CNF formula has the form

$$\bigwedge_i \left(\bigvee_j v_{ij} \right) \ ,$$

where each v_{ij} is either a variable u_k or to its negation $\neg u_k$. The terms v_{ij} are called the *literals* of the formula and the terms $(\vee_j v_{ij})$ are called its *clauses*. A kCNF is a CNF formula in which all clauses contain at most k literals.

The Cook-Levin Theorem

Theorem 2.10: Let SAT be the language of all satisfiable CNF formulae and 3SAT be the language of all satisfiable 3CNF formulae. Then,

- SAT is $\mathbf{NP}$-complete.
- 3SAT is $\mathbf{NP}$-complete.

Both languages are clearly in $\mathbf{NP}$. Thus we only need to prove that they are $\mathbf{NP}$-hard.

We do so by first proving that SAT is $\mathbf{NP}$-hard and then showing that SAT is polynomial-time Karp reducible to 3SAT. This implies that 3SAT is $\mathbf{NP}$-hard by the transitivity of polynomial-time reductions.

$\mathbf{NP}$-hardness of SAT

Thus, the following lemma is the key to the proof.

Lemma 2.12: SAT is $\mathbf{NP}$-hard.

To prove this we have to show how to reduce every $\mathbf{NP}$ language L to SAT, in other words give a polynomial-time transformation that turns any $x \in \{0,1\}^*$ into a CNF formula φ_x such that:

$$x \in L \iff \varphi_x \text{ is satisfiable } .$$

Since we know nothing about the language L except that it is in $\mathbf{NP}$, this reduction has to rely just upon the definition of computation, and express it in some way using a Boolean formula.

Warmup: Expressiveness of boolean formulae

Example 2.13: The formula $(a \vee \overline{b}) \wedge (\overline{a} \vee b)$ is in CNF form. It is satisfied by only those values of a, b that are equal. Thus, the formula

$$(x_1 \vee \overline{y}_1) \wedge (\overline{x}_1 \vee y_1) \wedge \ldots \wedge (x_n \vee \overline{y}_n) \wedge (\overline{x}_n \vee y_n)$$

is True if and only if the strings $x, y \in \{0, 1\}^n$ are equal to one another. Thus, though $=$ is not a standard boolean operator like $\vee$ or $\wedge$, we will use it as a convenient shorthand since the formula $\phi_1 = \phi_2$ is equivalent to (in other words, has the same satisfying assignments as) $(\phi_1 \vee \overline{\phi_2}) \wedge (\overline{\phi_1} \vee \phi_2)$.

CNF formulae of sufficient size can express every Boolean condition ...

Warmup: Expressiveness of boolean formulae

Claim 2.14: For every Boolean function $f\colon \{0,1\}^\ell \to \{0,1\}$ there is an ℓ-variable CNF formula φ of size $\ell 2^\ell$ such that $\varphi(u) = f(u)$ for every $u \in \{0,1\}^\ell$, where the size of a CNF formula is defined to be the number of $\wedge/\vee$ symbols it contains.

Proof Sketch: For every $v \in \{0,1\}^\ell$, it is not hard to see that there exists a clause C_v such that $C_v(v) = 0$ and $C_v(u) = 1$ for every $u \neq v$. For example, if $v = \langle 1,1,0,1\rangle$, the corresponding clause is $(\overline{u}_1 \vee \overline{u}_2 \vee u_3 \vee \overline{u}_4)$.

We let φ be the AND of all the clauses C_v for v such that $f(v) = 0$ (note that φ is indeed of size at most $\ell 2^\ell$). Then for every u such that $f(u) = 0$ it holds that $C_u(u) = 0$ and hence $\varphi(u)$ is also equal to 0. On the other hand, if $f(u) = 1$ then $C_v(u) = 1$ for every v such that $f(v) = 0$ and hence $\varphi(u) = 1$. We get that for every u, $\varphi(u) = f(u)$.

Proof of Cook-Levin: First idea

Let $L \in \mathbf{NP}$ and let M be the poly-time TM such that for every $x \in \{0,1\}^*$: $x \in L \Leftrightarrow \exists u \in \{0,1\}^{p(|x|)} M(x,u) = 1$, where p is a polynomial. We show that L is reducible to SAT by transforming in poly-time every string $x \in \{0,1\}^*$ into a CNF formula φ_x such that $x \in L$ iff φ_x is satisfiable.

How can we construct such a formula φ_x? By Claim 2.14, the function that maps $u \in \{0,1\}^{p(|x|)}$ to $M(x,u)$ can be expressed as a CNF formula ψ_x (i.e., $\psi_x(u) = M(x,u)$ for every $u \in \{0,1\}^{p(|x|)}$. Thus a string u such that $M(x,u) = 1$ exists iff ψ_x is satisfiable.

But this is not useful! The size of ψ_x is as large as $p(| x |)2^{p(|x|)}$.

To get a smaller formula we use the fact that M runs in polynomial time, and that each basic step of a Turing machine is *highly local* (in the sense that it examines and changes only a few bits of the machines tapes).

Oblivious Turing Machines

We will make the following simplifying assumptions about M:

- M has *two tapes*: an input tape and a work/output tape; and
- M is *oblivious*, i.e. its head movement does not depend on the contents of its input tape. In particular, this means that Ms computation takes the same time for all inputs of size n and for each time step i the location of Ms heads at the i-th step depends only on i and Ms input length. This is without loss of generality because:

Exercise: Prove that for every $T(n)$-time M there exists a two-tape oblivious $\tilde{M}$ computing the same function in $O(T^2(n))$ time.

Thus in particular, if L is in $\mathbf{NP}$ then there exists a two-tape oblivious polynomial-time M and a polynomial p such that:

$$x \in L \;\Leftrightarrow\; \exists u \in \{0,1\}^{p(|x|)} \colon M(x,u) = 1 \;\; .$$

Snapshots

As M is oblivious, we can define functions:

- $\texttt{inputpos}(i)$ – location of the input tape head at the i-th step
- $\texttt{prev}(i)$ – the last step before i that M visited the same location on its work tape. If there is no such step, then define $\texttt{prev}(i) = 0$

These functions can be computed in poly-time by simulating M on, say, the all-zeroes input.

Def: The *snapshot* z_i of $M(y)$ at step i is the triple $\langle a, b, q \rangle$, such that a and b are the symbols read by Ms heads from the two tapes and q is the state M is in at the i-th step.

$$z_0 = \langle y_0, \Box, q_{\text{start}} \rangle.$$

$$z_i = \langle y_{\texttt{inputpos}(i)}, \delta_w\left(z_{\texttt{prev}(i)}\right), \delta_q(z_{i-1}) \rangle, \qquad \text{if } i > 0.$$

Describing $M(y)$ with a Snapshot Sequence

The size of an encoded snapshot is $|z| = 1 + \log_2 |\Gamma| + \log_2 |Q|$.

For any M, there is a Boolean function F_M with $3 + 2\log_2 |\Gamma| + 2\log_2 |Q|$ input variables, such that for every i and for every input y:

$$z_i = F_M\left(y_{\texttt{inputpos}(i)}, z_{\texttt{prev}(i)}, z_{i-1}\right) . \qquad (1)$$

There is a Boolean function G_M with $4 + 3\log_2 |\Gamma| + 3\log_2 |Q|$ input variables that defines the condition (1):

$$G_M\left(y_{\texttt{inputpos}(i)}, z_{\texttt{prev}(i)}, z_{i-1}, z_i\right) = 1$$
$$\Leftrightarrow z_i = F\left(y_{\texttt{inputpos}(i)}, z_{\texttt{prev}(i)}, z_{i-1}\right) .$$

M is uniquely determined by G_M, because F contains both components δ_w, δ_q of M's transition function.

Proof of Cook-Levin: SAT is $\mathbf{NP}$-hard

For any oblivious M with running time $t(n)$ and $x \in \{0,1\}^n$, the condition $M(x,u) = 1$ (where $u \in \{0,1\}^{p(|x|)}$) can be uniquely described with an input string $y \in \{0,1\}^{n+p(n)}$ sequence of snapshots $z_0, z_1, z_2, \ldots, z_{t(n)}$, such that:

- First n bits of y encode x. Can be represented by a DNF with size $4n$
- $z_0 = \langle y_0, \Box, q_{\mathsf{start}} \rangle$. Can be trivially represented by a DNF with size $c2^c$, where c is the length of a snapshot.
- For every $i \in \{1, \ldots, t(n)\}$: $G_M \left(y_{\mathtt{inputpos}(i)}, z_{\mathtt{prev}(i)}, z_{i-1}, z_i \right) = 1$. Can be trivially represented by a DNF with size $t(n) \cdot (3c+1) \cdot 2^{3c+1}$
- $z_{t(n)} = \langle *, *, q_{\mathsf{accept}} \rangle$. Can be represented by a DNF with size $c2^c$

Hence, for every x, we have an efficiently computable Boolean formula $\varphi_x(y, z_0, \ldots, z_{t(n)})$, that is satisfiable if and only if there is $u \in \{0,1\}^{|x|+p(|x|)}$, such that $M(x,u) = 1$.

Proof of Cook-Levin: Reducing SAT to 3SAT

We will map a CNF formula φ into a 3CNF formula ψ, which is satisfiable iff φ is. We demonstrate first the case that φ is a 4CNF.

Let C be a clause of φ, say $C = u_1 \vee \overline{u}_2 \vee \overline{u}_3 \vee u_4$. We add a new variable z to the φ and replace C with the pair of clauses $C_1 = u_1 \vee \overline{u}_2 \vee z$ and $C_2 = \overline{u}_3 \vee u_4 \vee \overline{z}$. Clearly, if $u_1 \vee \overline{u}_2 \vee \overline{u}_3 \vee u_4$ is true then there is an assignment to z that satisfies both $u_1 \vee \overline{u}_2 \vee z$ and $\overline{u}_3 \vee u_4 \vee \overline{z}$ and vice versa: if C is false then no matter what value we assign to z either C_1 or C_2 will be false.

The same idea can be applied to a general clause of size 4, and in fact can be used to change every clause C of size k (for $k > 3$) into an equivalent pair of clauses C_1 of size $k - 1$ and C_2 of size 3 that depend on the k variables of C and an additional auxiliary variable z.

The web of reductions

Cook and Levin had to show how every $\mathbf{NP}$-language can be reduced to SAT.

To prove the $\mathbf{NP}$-completeness of any other language L, we do not need to work as hard: it suffices to reduce SAT or 3SAT to L. Once we know that L is $\mathbf{NP}$-complete we can show that an $\mathbf{NP}$-language L_0 is in fact $\mathbf{NP}$-complete by reducing L to L_0.

This approach has been used to build a *web of reductions* and show that thousands of interesting languages are in fact $\mathbf{NP}$-complete.

INDSET is $\mathbf{NP}$-complete

Proof: Since INDSET is clearly in $\mathbf{NP}$, we only need to show that it is $\mathbf{NP}$-hard, which we do by reducing 3SAT to INDSET. Let φ be a 3CNF formula on n variables with m clauses. We define a graph G of $7m$ vertices as follows: we associate a cluster of 7 vertices in G with each clause of φ.

The vertices in cluster associated with a clause C correspond to the 7 possible partial assignments to the three variables C depends on (they are *partial assignments*, since they only give values for some of the variables).

If C is $\overline{u}_2 \vee \overline{u}_5 \vee \overline{u}_7$ then the 7 vertices in C correspond to all partial assignments of the form $u_1 = a$, $u_2 = b$, $u_3 = c$ for a binary vector $\langle a, b, c \rangle \neq \langle 1, 1, 1 \rangle$. (If C depends on less than three variables then we

repeat one of the partial assignments and so some of the 7 vertices will correspond to the same assignment.)

We put an *edge* between two vertices of G if they correspond to *inconsistent* partial assignments. Two partial assignments are *consistent* if they give the same value to all the variables they share. For example, the assignment $u_1 = 1, u_2 = 0, u_3 = 0$ is inconsistent with the assignment $u_3 = 1, u_5 = 0, u_7 = 1$ because they share a variable (u_3) to which they give a different value.

In addition, we put edges between every two vertices that are in the same cluster.

Clearly transforming φ into G can be done in poly-time.

We claim that φ is satisfiable iff G has an independent set of size m. Indeed, suppose that φ has a satisfying assignment u. Define a set S of m vertices as follows: for every clause C of φ put in S the vertex in the cluster associated with C that corresponds to the restriction of u to the variables C depends on. Because we only choose vertices that correspond to restrictions of the assignment u, no two vertices of S correspond to inconsistent assignments and hence S is an independent set of size m.

On the other hand, if G has an independent set S of size m then define a satisfying assignment u as follows: for every $i \in \{1, \ldots, n\}$, if there is a vertex in S whose partial assignment gives a value a to u_i, then set $u_i = a$; otherwise set $u_i = 0$. This is well defined because S is an independent set, and hence each variable u_i can get at most a single value by assignments of the vertices in S. On the other hand, S can contain at most one vertex in each cluster, and hence there is an element of S in every one of the m clusters. Thus, by the definition u, it satisfies all φs clauses.

coNP

Def: $\mathbf{coNP} = \{L \subseteq \{0,1\}^* \colon \overline{L} \in \mathbf{NP}\}$.

Hence, $\overline{\text{SAT}} \in \mathbf{coNP}$.

Alternative def: $L \in \mathbf{coNP}$ if there exists a polynomial p and a polynomial-time Turing machine M such that for every $x \in \{0,1\}^*$:

$$x \in L \Leftrightarrow \forall u \in \{0,1\}p(|x|) \colon M(x,u) = 1 \ .$$

TAUTOLOGY is $\mathbf{coNP}$-complete

In classical logic, tautologies are true statements. The following language is $\mathbf{coNP}$-complete:

$$\mathsf{TAUTOLOGY} = \{\varphi\colon\ \varphi \text{ – Boolean formula that is satisfied by every assignment}\}\ .$$

It is clearly in $\mathbf{coNP}$ and so all we have to show is that for every $L \in \mathbf{coNP}$, $L \leq_p$ TAUTOLOGY. But this is easy: just modify the Cook-Levin reduction from $\overline{L}$ (which is in $\mathbf{NP}$) to SAT. For every input $x \in \{0,1\}^*$ that reduction produces a formula φ_x that is satisfiable iff $x \in \overline{L}$. Now consider the formula $\neg\varphi_x$. It is in TAUTOLOGY iff $x \in L$, and this completes the description of the reduction.

$\mathbf{EXP}$ and $\mathbf{NEXP}$

The following two classes are exponential time analogues of $\mathbf{P}$ and $\mathbf{NP}$.

Def:

- $\mathbf{EXP} = \cup_{c \geq 0} \mathbf{DTIME}(2^{n^c})$.
- $\mathbf{NEXP} = \cup_{c \geq 0} \mathbf{NTIME}(2^{n^c})$.

Because every problem in $\mathbf{NP}$ can be solved in exponential time by a brute force search for the certificate, $\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{EXP} \subseteq \mathbf{NEXP}$.

Is there any point to studying classes involving exponential running times?

The following simple result may be a partial answer.

If $\mathbf{EXP} \neq \mathbf{NEXP}$ then $\mathbf{P} \neq \mathbf{NP}$

We prove the contrapositive: $\mathbf{P} = \mathbf{NP}$ implies $\mathbf{EXP} = \mathbf{NEXP}$.

Suppose $L \in \mathbf{NTIME}(2^{n^c})$ and NDTM M decides it. We claim that then the language

$$L_{\text{pad}} = \{\langle x, 1^{2^{|x|^c}}\rangle : x \in L\}$$

is in $\mathbf{NP}$. Here is an NDTM for L_{pad}:

- given y, first check if there is a string z such that $y = \langle z, 1^{2^{|z|^c}}\rangle$. If not, output REJECT.
- If y is of this form, then run M on z for $2^{|z|^c}$ steps and output its answer.

Clearly, the running time is polynomial in $\mid y \mid$, and hence $L_{\text{pad}} \in \mathbf{NP}$. Hence if $\mathbf{P} = \mathbf{NP}$ then L_{pad} is in $\mathbf{P}$. But if L_{pad} is in $\mathbf{P}$ then L is in $\mathbf{EXP}$: to determine whether an input x is in L, we just pad the input and decide whether it is in L_{pad} using the polynomial-time machine for L_{pad}.

What have we learned?

- The class $\mathbf{NP}$ consists of all the languages for which membership can be certified to a polynomial-time algorithm. It contains many important problems not known to be in $\mathbf{P}$. $\mathbf{NP}$ can also be defined using non-deterministic Turing machines.

- $\mathbf{NP}$-complete problems are the hardest problems in $\mathbf{NP}$, in the sense that they have a polynomial-time algorithm if and only if $\mathbf{P} = \mathbf{NP}$. Many natural problems that seemingly have nothing to do with Turing machines turn out to be $\mathbf{NP}$-complete. One such example is the language 3SAT of satisfiable Boolean formulae in 3CNF form.

- If $\mathbf{P} = \mathbf{NP}$ then for every search problem for which one can efficiently verify a given solution, one can also efficiently find such a solution from scratch.